

SPIN-FLAVOUR SYMMETRY AND CONTRACTIONS TOWARDS CLASSICAL SPACE-TIME SYMMETRY

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A classification scheme of hadrons is proposed on the basis of the division algebra \mathbf{H} of quaternions and an appropriate geometry. This scheme suggests strongly to understand flavour symmetry in another manner than from standard symmetry schemes. In our approach, we do *not* start from ‘exact’ symmetry groups like $SU(2) \times SU(2)$ chiral symmetry and impose various symmetry breaking mechanisms which collide with theorems wellknown from quantum field theory. On the contrary, the approximate symmetry properties of the hadron spectrum at low energies, usually classified by ‘appropriately’ broken compact flavour groups, emerge very naturally as a low energy reduction of the noncompact (dynamical) symmetry group $Sl(2, \mathbf{H})$. This quaternionic approach not only avoids most of the wellknown conceptual problems of Chiral Dynamics but it also allows for a general treatment of relativistic flavour symmetries as well as it yields a direct connection towards classical relativistic symmetry.

1. Introduction

The standard approach towards a classification scheme of hadrons is still founded on Heisenberg’s¹ very old idea that the relevant flavour degrees of freedom can be described by a $SU(2)$ symmetry group (‘Isospin’) and on Yukawa’s² hypothesis of a (quantized) meson exchange to mediate nuclear forces. In the mean time, a lot of more sophisticated mechanisms have been added to this very basic concept to approach or even understand the plenty of available experimental data. However, to explain these data on the basis of $SU(2)$ flavour or $SU(2) \times SU(2)$ chiral theory almost all of the added ideas and mechanisms introduce further theoretical difficulties and sometimes even serious deficiencies.

In the following, we’ll start with a brief outline of symmetry methods where we focus especially on the manner of corrections applied to $SU(2)$ isospin symmetry. Afterwards, we’ll discuss a new symmetry approach which avoids these conceptual problems from the very beginning in that we do *not* start from a compact (flavour) symmetry group and try to relate it constructively to space-time symmetry. On the contrary, we understand compact flavour or chiral symmetry (which anyhow are appropriate descriptions only in the very low energy regime of the hadronic spectrum) as reasonable low energy approximation schemes of ‘the real’ dynamical

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symmetry group $\text{Sl}(2, \mathbf{H})$. This noncompact symmetry group can be found by pure geometrical considerations as well as by trying to unify all the physical facts known from flavour, chiral and Wigner supermultiplet theory³. We present a brief review of the resulting algebraic theory^{4 5} based on the Lie algebra isomorphism $\text{sl}(2, \mathbf{H}) \cong \text{su}^*(4) \cong \text{so}(5, 1)$ before we focus on the route towards classical space-time symmetry.

2. Flavour symmetry

At the beginning of the 60ies, field theoretical investigations of pions and pion-nucleon interactions⁶ at low energies on the basis of pure $\text{SU}(2)$ isospin interactions led to results which were different from available data. Although the $\text{SU}(2)$ pseudoscalar coupling scheme is suggested by a first sight on ratios of cross sections as well as by a naive comparison of $\text{SU}(2)$ representations with the particle structure in the low energy regime of the hadron spectrum, it doesn't reproduce neither the pseudovectorial coupling of the pion to the nucleon nor the particle structure of the spectrum at higher energies nor transitions between $\text{SU}(2)$ multiplets without introducing additional free parameters. Weinberg's pioneering work^{7 8} on the pseudovectorial pion-nucleon coupling and more general investigations on effective Lagrangians^{9 10 11} allowed to understand the pion coupling to the nucleon on a new footing with an underlying $\text{SU}(2) \times \text{SU}(2)$ symmetry group, denoted by 'Chiral Dynamics'. Although the larger chiral group respected the parity independence of strong interactions in addition to its charge independence (isospin) and although the chiral approach led to a reasonable description of the low-energy pion-nucleon coupling properties in terms of effective field theory, its strict interpretation has several serious deficiencies:

- The pseudovectorial coupling $\partial_\mu \vec{\pi}$ of the pion destroys renormalizability of the Lagrangian in perturbation theory and thus requires a *new* interpretation of Lagrangians in terms of 'effective Lagrangians' as well as a redefinition of the applied methods termed to as 'effective field theory'.
- In the hadron spectrum, $\text{SU}(2) \times \text{SU}(2)$ representations of Chiral Dynamics are not realized via the Wigner-Weyl mode. To work around this defect, the concept of a spontaneously broken symmetry has been introduced to realize $\text{SU}(2)$ isospin quantum numbers, and the three pion fields were interpreted as the necessary Goldstone bosons of this picture. However, Goldstone bosons have to be massless particles so that the observable mass of the pion triplet has to be restored by the further assumption of an additional explicit symmetry breaking mechanism, usually parametrized by partially conserved axial currents (PCAC theorem). Please note, however, that all these additional assumptions on symmetry breaking as well as on an identification of group representations are appropriate methods only when dealing with compact symmetry groups and a welldefined perturbation theory. In this case, all possible group actions connect only the finite number of states organized within the same irreducible representations of the group. A typical, very nice and simple

example is quantum mechanics where the elements of the perturbation series

$$S_{fi} \sim \langle f | \mathcal{H} | i \rangle = \langle f | H + \epsilon H' | i \rangle \quad (1)$$

can be understood well in terms of group theory. For compact groups, it is possible to use finite representation spaces according to properties of $|i\rangle$ and $|f\rangle$, to classify the symmetry breaking part H' of the Hamiltonian \mathcal{H} according to its covariance properties under symmetry transformations which leave the unperturbed Hamiltonian H invariant, and to define appropriate measures on the representation spaces in order to calculate (1). In addition, one may reparametrize H in terms of an exponential of the Lie algebra, use theorems on completeness of matrix elements, etc. For compact groups, the concepts of spontaneously and explicitly broken symmetries can be understood within a simple geometrical framework¹². However, in the context of a noncompact group or of $SU(2) \times SU(2)$ Chiral Dynamics, a naive transfer of these symmetry concepts raises severe problems. Besides the fact that an appropriate representation theory as well as a suitable measure theory becomes very intricate or isn't even available, it has been shown for spontaneously broken symmetries that some of the symmetry generators ('charges') do not exist on the representation spaces of a (compact) subgroup as well as it makes no sense to argue about the magnitude of symmetry breaking^{13 14 15}. Thus, dealing with $SU(2) \times SU(2)$ Chiral Dynamics and identifying physical particles with $SU_V(2)$ (isospin) representations, it is obvious that axial transformations mix different (irreducible) isospin representations ('superselection rules'), i.e. they mix the physical particle states of this picture (e.g. $\sigma \leftrightarrow \vec{\pi}$, $N \leftrightarrow \Delta$). Furthermore, it is *not possible* to represent the axial charges in terms of one-particle isospin states, and the discussion of a continuous limit $m_\pi^2 \rightarrow 0$ of the symmetry breaking parameter has no theoretical foundation as a perturbative approximation of the welldefined 'symmetry limit' $m_\pi^2 = 0$.

- With respect to a relativistic symmetry formulation and the 'No-Go-Theorems'⁸, it is interesting to note that Chiral Dynamics obviously suggests a non-standard treatment of relating compact to noncompact (dynamical) symmetry groups. The symmetry scheme underlying Chiral Dynamics is far from being a supersymmetric one, and it also doesn't decouple compact and noncompact symmetry transformations via a direct product structure of the groups. On the contrary, a comparison of Chiral Dynamics with low energy data strongly suggests to couple the vector in isospin space with vectorial (p-wave) transformation properties under orbital angular momentum transformations^a \vec{L} .

^aUsing Weyl's unitary trick, it is straightforward to transfer the transformation properties of \vec{L} with respect to homogeneous Lorentz transformations to the compact analogon $SU(2) \times SU(2)/SU(2)$ which is isomorphic to the symmetry scheme of Chiral Dynamics. Within the related noncompact and compact schemes, the diagonal subgroup $SU_V(2)$ may be identified with spatial rotational symmetry and with isospin, respectively, and the available low energy data strongly suggests to identify the pion fields in both schemes (isovector/pseudovector) within the adjoint representation

In the following, we present physical arguments as well as the algebraic foundation of our new ansatz which yields a lot of known symmetry properties of hadrons.

3. Dynamic flavour symmetry

3.1. *Physical motivation*

Referring to Sudarshan's investigations¹⁶ on spontaneous symmetry breaking and approximate symmetries of hadrons, we have shown^{4 12} that various approaches to classify hadron multiplets in the low energy regime of the spectrum lead to an effective SU(4) theory. This effective symmetry scheme has the further advantage to yield a good description of hadron transformation properties, however, like in the case of Wigner's supermultiplet theory of nuclear ground states we are faced with an approximate (compact) symmetry group which yields a reasonable low energy description and describes a lot of observable symmetry properties but becomes worse at higher energies. This dynamical similarity in the behaviour of completely different physical systems motivates the viewpoint that the compact symmetry group SU(4) respectively its compact subgroups SU(2)×SU(2), SU(2) and the structure SU(2)×SU(2)/SU(2) known from flavour symmetry are low energy approximations of a suitable noncompact (dynamical) symmetry group. Using Weyl's unitary trick, we find SU*(4) which results from representing quaternions on complex vector spaces and which has the compact symmetry group Sp(2) ≅ USp(4) as maximal compact subgroup in common with SU(4).

3.2. *Mathematical motivation*

A straightforward mathematical approach is guided by the necessity of a spinorial calculus to describe the observable half-integer spin/isospin representations suitably. The simplest case of spinorial calculus emerges in the framework of the stereographic projection $S^2 \rightarrow \mathbf{R}^2$. To map all closed paths on S^2 (especially those passing through the north pole of the sphere!) appropriately into the plane it is necessary to introduce one hypercomplex unit i , i.e. to complexify the planar cartesian coordinates relative to each other. Thus, we may use either a complex planar variable z and its complex conjugate or two real angles denoting points on the sphere. If we introduce in addition two homogeneous complex coordinates z_1 and z_2 related to z via $z = z_1/z_2$, we can investigate Möbius transformations of z ,

$$f : z \rightarrow f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbf{C}, \quad (2)$$

or an equivalent Sl(2, C) matrix and spinor formalism which allows to apply the formalism of groups and appropriate representation theory. Thus, several different mathematical description are available for one and the same geometrical picture.

of the diagonal subgroup. This behaviour is in excellent agreement with the viewpoint to understand flavour symmetry as a low energy approximation of an appropriate (noncompact) dynamical symmetry group.

An appropriate generalization of the spinorial concept to four dimensions, however, is *not* based on the group theoretical description. Instead, eq. (2) gives all the necessary hints if we study the projection $S^4 \rightarrow \mathbf{R}^4$. To close paths through the ‘north pole’, it is necessary to complexify the coordinates of \mathbf{R}^4 pairwise. Furthermore, in eq. (2) multiplication as well as addition has to be defined between the ‘numbers’, and inverse elements of the denominator have to exist. In order to choose the appropriate mathematical tools it is thus necessary to look at least for an algebra. Moreover, the function f should have a *unique* singularity ‘to project the north pole to infinity’. This condition forbids zero divisors and thus restricts the possible choices from general algebras to the four division algebras with unit element. In the four dimensional case, we are thus left with Hamilton’s quaternions. However, to avoid problems with quaternionic analysis we do not use generalized (quaternionic) Möbius transformations $f(q)$ but the related matrix group $\text{Sl}(2, \mathbf{H})$ as well as $\text{SU}^*(4)$ and $\text{SO}(5,1)$. The infinitesimal properties of these symmetry transformations are accessible via the Lie algebra $\mathfrak{sl}(2, \mathbf{H})$ and will be investigated in the next section on quaternionic, on complex and on real representation spaces due to the Lie algebra isomorphism $\mathfrak{sl}(2, \mathbf{H}) \cong \mathfrak{su}^*(4) \cong \mathfrak{so}(5,1)$. Vice versa, Lie theory allows to integrate the infinitesimal symmetry transformations.

4. Algebraic approach to hadrons

4.1. $\text{Sl}(2, \mathbf{H})$ and $\text{SU}^*(4)$

The regular representation of $\text{Sl}(2, \mathbf{H})$ is isomorphic to the Dirac algebra⁵ and thus provides the mathematical framework of quantum field theory whereas $\text{SU}^*(4)$ justifies to use $\text{SU}(4)$ as an effective low energy approximation of the dynamical structure of the hadron spectrum^{5 12 4}. $\text{SU}(4)$ allows to identify nucleons and delta resonances within its spinorial third rank symmetric representation **20** and the mesons π , ω and ρ within the spinorial second rank representation **15**^b. The algebra used in Chiral Dynamics is completely contained within $\text{SU}(4)$, and it is possible to discuss flavour and chiral transformation properties of hadrons in terms of $\text{SU}(4)$ representations realized in the Wigner-Weyl mode⁴. It is noteworthy that the symmetry scheme $\text{Sl}(2, \mathbf{H})/\text{Sp}(2)$ resp. its complex representation $\text{SU}^*(4)/\text{USp}(4)$ allows to identify F. Klein’s ideas on complexified quaternions. Further investigations of Dirac theory in terms of complexified quaternions show that the mass is an arbitrary parameter which drops out completely, and that we are left with a theory using velocities as basic parameters^{12 5}. This leads to a Lobachevskian geometry and a relation between parameters of the Lie algebra $\mathfrak{sl}(2, \mathbf{H})$ (or $\mathfrak{su}^*(4)$) and *velocities*⁵ (see also¹⁷).

^bScreening the projective background of quaternionic spinors, the complex spinorial representations of mesons and nucleons/deltas suggest directly the ‘quark substructure’ of hadrons, i.e. a threefold complex spinorial index for nucleon and delta degrees of freedom and a ‘dotted’/‘undotted’ pair of indices to denote the vectorial properties of the meson representation. Vice versa, a theory based on complex vector spaces *has* to reproduce a threefold spinorial index to describe hadronic fermions and a pair of conjugated indices to describe the mesons thus respecting the quaternionic foundations.

4.2. $SO(5,1)$ and Classical Space-Time

The Lie algebra $so(5,1)$ contains the de Sitter algebra $so(4,1)$ which can be Wigner-Inönü-contracted¹⁸ towards the Poincaré algebra and to the Galilei algebra in the limits of vanishing curvature ($R \rightarrow \infty$) of the universe and vanishing ratio v/c ($c \rightarrow \infty$), respectively¹⁹. Here, we don't review all the technical details but it is noteworthy that none but the algebras $so(4,1)$ and $so(3,2)$ may be contracted to the Poincaré algebra²⁰. It is a second important property of contractions that widely used differential operators like ∂_μ appear only *after* contraction of the generators of Lie groups. Appropriately, this description is isomorphic to projective ('flat') physics where the operators ∂_μ represent contracted elements of the full Lie algebra $so(4,1)$ which itself spans a vector space, the tangent space to the Lie group $SO(4,1)$ at unity. As a direct consequence, the definition of observable 'mass' is related *only* to the contracted limit of the more general Lie algebra $so(4,1)$ so that there is no foundation to introduce mass parameters in the framework of group transformations acting on homogeneous coordinates. This suggests to understand mass as a classical effective parameter which emerges after the contraction process, and it explains why the mass parameters drop out when using complexified quaternions to describe Dirac theory. In addition, if we interpret interactions in Dirac theory on the basis of $sl(2, \mathbf{H})$ transformations acting on homogeneous quaternionic coordinates, these interactions necessarily change the 'masses' of the involved particles.

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